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Models for Disc Infiltrometers

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Disc infiltrometers are popular devices for determining in situ hydraulic properties of unsaturated soils. This paper compares alternative solutions of Richards' equation for both steady state and time-dependent cases. The steady state solutions were in general agreement using alternative hydraulic conductivity functions of the same capillary length scale. Small-time solutions for the nonlinear cases were consistent with linear diffusion from a disc source using an average diffusivity value which is simply related to the capillary length. This offers a refinement over the one-dimensional solution for short times in that the geometric effect of the circular source is included. Simulations for two examples indicate that the approach to the steady state solution may take considerably longer than what is commonly reported in the literature for field applications.

INTRODUCTION

Disc infiltrometers are becoming increasingly popular for determining in situ hydraulic properties of soil [cf. *Clothier and White*, 1981; *White and Sully*, 1988; *Ankeny et al.*, 1988, 1991; *Smettem and Clothier*, 1989; *Reynolds and Elrick*, 1991]. They are designed to measure intake at a carefully controlled water pressure within a circular interface at the soil surface. The water pressure can be slightly positive, but more often is at a small tension of about 0–0.2 m of water. By maintaining the entry head at a tension, flow into the larger macropores can be avoided.

The analysis of data from disc infiltrometers has rested heavily on the following five equations. The first is an approximation for early times and assumes equivalence to a one-dimensional system [cf. *White and Sully*, 1987, 1988]:

$$\frac{Q}{\pi r_0^2} \approx 0.5\ S t^{-0.5} \qquad (1)$$

where Q is the flow from the disc infiltrometer (in m^3 s^{-1}), r_0 is the disc radius (in meters), S the sorptivity (in m s$^{-0.5}$), and t time (in seconds). The sorptivity is dependent upon the initial water content, the supply water content, and the diffusivity function and is an integral measure of soil capillarity [*Philip*, 1955, 1969]. Integration of (1) with respect to t results in the cumulative intake per unit area I (in meters).

$$I = S t^{0.5} \qquad (2)$$

For small times, S is simply the slope of I versus $t^{0.5}$.

The third commonly used relationship is for the macroscopic capillary length λ_c [*Philip*, 1985; *White and Sully*, 1987, 1988], defined by

$$\lambda_c = [K_{\text{wet}} - K_{\text{dry}}]^{-1} \int_{h_{\text{dry}}}^{h_{\text{wet}}} K(h)\, dh \qquad (3)$$

where K_{wet} and K_{dry} are the conductivity values corresponding to the supply matric potential h_{wet} and the initial matric potential h_{dry}. The approximate relationship of λ_c to S is

$$\lambda_c = \frac{bS^2}{(\theta_{\text{wet}} - \theta_{\text{dry}})(K_{\text{wet}} - K_{\text{dry}})} \qquad (4)$$

This was proposed by *White and Sully* [1987, 1988], with θ_{wet} the supply water content and θ_{dry} the initial value. They considered three hydraulic functions. *Warrick and Broadbridge* [1992] confirmed that $b = 0.55$ is a representative value using four hydraulic functions in addition to those of *White and Sully* [1987]. One requirement for equivalence between parameters of alternative hydraulic functions, as used by *Russo et al.* [1991], is that S be the same for equal θ_{wet}, θ_{dry}, and K_{wet}. This will be the case if λ_c/b is equal, as can be verified from (4) with $K_{\text{wet}} \gg K_{\text{dry}}$.

Finally, there is the approximation for flow from a disc at steady state conditions [*Wooding*, 1968]:

$$Q \approx \pi r_0^2\, K_{\text{wet}} \left[1 + \frac{4\lambda_c}{\pi r_0}\right] \qquad (5)$$

where Q is the flow volume per unit time (in m^3 s^{-1}) and K_{wet} is the hydraulic conductivity value corresponding to the water supply. Wooding's solution was based on a hydraulic conductivity of the form [*Gardner*, 1958]

$$K = K_s \exp(\alpha h) \qquad (6)$$

with h (in meters) the matric potential, K_s (in meters per second) a constant normally taken as the saturated hydraulic conductivity, and α (in m^{-1}) a constant equivalent to λ_c^{-1}. A refinement of Wooding's equation for small αr_0 was offered by *Weir* [1987]. Consideration of (2), (4), and (5) leads to a solution for K_{wet}, provided K_{dry} is negligible and that θ_{wet} and θ_{dry} are measured.

An alternative approach based only on Wooding's solution is possible when Q values are known for two or more r_0 values [*Scotter et al.*, 1982; *Yitayew and Watson*, 1986; *Smettem and Clothier*, 1989; *Hussen*, 1991]. This allows K_{wet} and λ_c to be found directly by solving two equations of the form of (5). If results for three or more r_0 values are available, then K_{wet} and λ_c can be evaluated using a "best fit" procedure. The same principle can be applied for a single r_0 value with multiple tensions [*Lien*, 1989; *Ankeny et al.*, 1991; *Hussen*, 1991].

An alternative but necessarily more complex approach can be based on numerical simulations. Numerous algorithms have been used to model transient movement into a

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TABLE 1. Hydraulic Functions Considered in the Examples

	K/K_s	Θ
Gardner [1958] and *Russo* [1988]	$\exp h^*$	$[\exp(h^*/2)(1 - h^*/2)]^{2/(m+2)}$
van Genuchten [1980]	$\Theta^{0.5}[1-(1-\Theta^{1/m})^m]^2$	$[1 + \lvert h^*\rvert^{1/(1-m)}]^{-m}$

Note that m is a generic parameter and not specific to a single hydraulic function.

uniformly dry soil from point or localized water sources. For trickle irrigation, finite difference solutions were given by *Brandt et al.* [1971] and *Lafolie et al.* [1989]; *van der Ploeg and Benecke* [1974] and *Fletcher-Armstrong and Wilson* [1983] used the continuous simulation modeling program; and *Taghavi et al.* [1984, 1985] applied finite elements. There are also more traditional groundwater models, several of which are reviewed by *Lappala et al.* [1987] and *McKeon and Chu* [1987].

The objective of the study is to compare alternative solutions for disc infiltrometers. First the utility of the *Wooding* [1968] and *Weir* [1987] solutions and the capillary length concept will be tested using an alternative conductivity function [*van Genuchten*, 1980]. Next the small-time solution for diffusion by *Chu et al.* [1975], as extended to a disc by *Warrick et al.* [1992], will be tested against nonlinear simulations. Finally, large-time solutions will be examined to address the question of the time required for infiltration from a disc to approach steady state.

THEORY

The Richard's equation describes unsaturated water flow as

$$\frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rK\frac{\partial h}{\partial r}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) - \frac{\partial K}{\partial z} \tag{7}$$

where θ is the volumetric water content, h is matric potential (in meters), r is a radial coordinate, z is depth, t is time, and K is the unsaturated hydraulic conductivity.

Appropriate initial and boundary conditions for the tension infiltrometer are

$$h(r, z, 0) = h_{\text{dry}} \tag{8}$$

$$h(r, 0, t) = h_{\text{wet}} \qquad 0 < r < r_0 \tag{9}$$

$$-\frac{\partial h}{\partial z} + 1 = 0 \qquad z = 0 \qquad r > r_0 \tag{10}$$

$$h(r, z, t) = h_{\text{dry}} \qquad r^2 + z^2 \to \infty \tag{11}$$

The solution of (7) subject to (8)–(11) will be a function of r, z, and t as well as α, h_{dry}, and h_{wet}.

The total surface flux is of particular interest for the disc permeameter. This may be expressed by

$$Q = 2\pi K_{\text{wet}} \int_0^{r_0} \left(1 - \left.\frac{\partial h}{\partial z}\right|_{z=0}\right) r\, dr \tag{12}$$

or

$$Q = \pi r_0^2 K_{\text{wet}}\left(1 - \left\langle\frac{\partial h}{\partial z}\right\rangle\right) \tag{13}$$

with $\langle \partial h/\partial z\rangle$ the average value of the pressure gradient at the disc surface.

The two hydraulic functions listed in Table 1 will be used. In both cases dimensionless matric potentials and water contents are defined as

$$h^* = \alpha h \tag{14}$$

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} \tag{15}$$

where α is now a generalized soil dependent constant (in m^{-1}), θ_r is a "residual" water content, and θ_s is the saturated water content. The first set is for the hydraulic conductivity function of *Gardner* [1958] with the soil water characteristic of *Russo* [1988]. For these the α is simply $1/\lambda_c$ by (3). The second set of functions is by *van Genuchten* [1980]. The correspondence of α to λ_c in this case is implicitly given by (3). Some values are presented in Table 3 following *Warrick and Broadbridge* [1992]. The product $\alpha\lambda_c$ is a function of m, αh_{dry}, and αh_{wet} and is generally less than 1 (Table 3).

For the above hydraulic functions, reduced forms of Richard's equation are possible in terms of dimensionless parameters. The mean gradient $\langle \partial h/\partial z\rangle$ of (13) is independent of α provided αh_{wet}, αh_{dry}, and αr_0 are the same. This is particularly useful in that the dimensionless surface flux Q^* given by

$$Q^* = \alpha^2 Q/K_s \tag{16}$$

will be invariant for steady state conditions provided also that αh_{wet}, αh_{dry}, and αr_0 are the same. If αh_{dry} is very small (approaching $-\infty$) and αh_{wet} is 0, the Q^* will be the same provided αr_0 is the same. For example, Q^* for $\alpha = 1$ and $r_0 = 0.1$ would be the same as for $\alpha = 0.1$ and $r_0 = 1$. This helps generalize the flow rates which will be presented shortly.

Chu et al. [1975] considered heat conduction from a half plane with the remaining half plane thermally insulated. If we consider a cartesian coordinate system, this is equivalent to a boundary in the $y = 0$ plane over which a constant potential is maintained for $x > 0$ and which is insulated for $x < 0$. The initial condition is a constant potential (different than the boundary potential), and the thermal diffusivity is constant. *Vauclin et al.* [1977] compared the Chu et al. solution with a numerical solution for two dimensions. *Warrick et al.* [1992] developed an approximation for linear diffusion from a disc source by combining these results with an asymptotic form for large values of time. A similar approximation, discussed in the appendix, is

$$Q(T) = 2\pi^{0.5} S D^{0.5} r_0 [1 + 0.25\pi^{0.5} T^{-0.5} - 0.216 \exp(-4.01T)] \tag{17}$$

where

$$T = Dt/r_0^2 \quad (18)$$

The approximation is valid for all times for linear diffusion, with D(in $m^2\ s^{-1}$) a constant diffusion coefficient. For small times the above expression is equivalent to

$$\frac{Q}{\pi r_0^2} \approx 0.5\ St^{-0.5} + 0.885\ \frac{SD^{0.5}}{r_0} \quad (19)$$

and is closely related to the results of *Chu et al.* [1975] and of *Warrick et al.* [1992, equation 34]. Although similar to the first two terms of *Philip*'s [1969] solution for one-dimensional infiltration, here the second term is due only to geometrical affects and is independent of gravity.

The above relationships suggest an empirical equation for infiltration from the disc as

$$I = St^{0.5} + \frac{Q_{ss}}{\pi r_0^2}\left\{t + \frac{c}{d}\left[1 - \exp(-dt)\right]\right\} \quad (20)$$

where c and d are empirical constants and Q_{ss} is the steady state value. This relationship has been used successfully to fit field data by *Hussen* [1991], who called it a "modified Horton" equation.

NUMERICAL CALCULATIONS

In order to carry out necessary numerical calculations, a finite element program "Disc" was developed. The finite element formulation was chosen to allow ease and flexibility for assigning boundary conditions and the spatial fineness of the numerical approximations. The procedure used the Galerkin method of weighted residuals and closely followed the programs developed by *Silva* [1990] for infiltration from an irrigation furrow. A mixed formulation was used for water content and pressure head, as suggested by *Celia and Bouloutas* [1990]. They found that mass conservation was generally easier to maintain than when using a purely pressure-head formulation. The algorithm can be run in either a time-dependent or steady state mode.

Initial and boundary conditions along the soil surface are given by (9) and (10). Along the z axis (DE of Figure 1) the boundary condition is that of no lateral flow. The other two boundaries (AB and AE of Figure 1) are more difficult because they are formally defined only as $r^2 + z^2$ approaches infinity. These were approximated by

$$\partial h/\partial r = 0 \qquad r = r_{max} \quad (21)$$

$$\partial h/\partial z = 0 \qquad z = z_{max} \quad (22)$$

This assumes that r_{max} is chosen large enough that lateral flow at $r = r_{max}$ is negligible. Also, z_{max} must be large enough that a unit hydraulic gradient is approximated, although h can vary horizontally along AE.

The generation of nodes was chosen on a variably spaced grid. The horizontal spacing was fixed for $0 < r < r_0$ (beneath the disc) and then varied by an arithmetic progression between r_0 and r_{max}. The number of nodes on the disc was specified, as well as the total number of horizontal nodes. The vertical nodes were chosen as an arithmetic progression by specifying the minimal spacing, the number of vertical nodes, and z_{max}. This defined all the nodes for the flow region, after which triangular elements were formed by drawing diagonals across the rectangles formed by the nodes. The program was coded in FORTRAN 77.

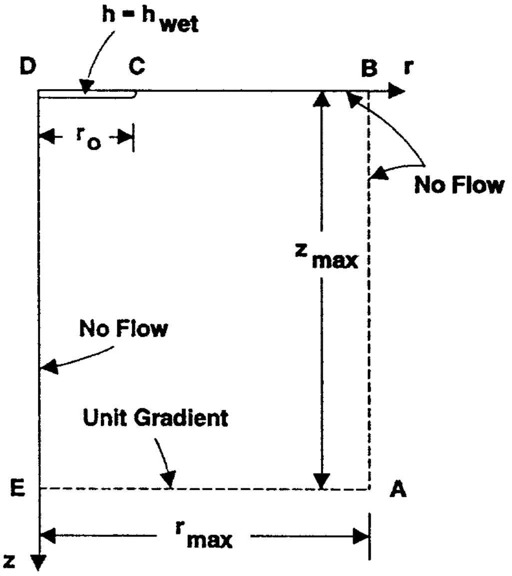

Fig. 1. Geometry considered for the disc infiltrometer. The source is along DC.

Generic errors of the numerical approximations were considered as follows:

1. Regarding mass continuity, for the time-dependent cases, water added was tested against the change in storage and water drained. Typically, the discrepancy was less than 0.1% of the water added. For the steady state examples the rates of water added and drained were consistent within three significant digits.

2. For spatial discretization the strategy was to use successively greater numbers and smaller elements until inflow amounts did not change. Similarly, successively smaller time steps were chosen until results did not change appreciably.

3. Errors arise from assuming a finite flow regime [(21) and (22)]. The value of r_{max} was varied with the results compared to *Wooding* [1968] and *Weir* [1987] as part of example 1 below. Also, comparisons using (21) to those with $h = -10$ and -100 m at $r = r_{max}$ were made. For $\alpha r_0 = 0.01$, r_{max}/r_0 should be large (such as 50); for $\alpha r_0 = 1$, r_{max}/r_0 can be smaller (such as 5).

Other calculations included those for finding the sorptivity and scaling length λ_c. The sorptivity was defined by *Philip* [1955]. For the integration of the conductivity function of (3) the conductivity form was used for the wet range ($0.95 < \Theta < 1$). For the drier range ($\Theta < 0.95$) the equivalent diffusivity form was used:

$$\int K(h)\ dh = \int D(\theta)\ d\theta \quad (23)$$

This avoids undefined values for D as θ approaches saturation and undefined h values as h approaches $-\infty$.

Example 1: Comparison of Steady State Values

As an overall test of "Disc," element size, r_{max}, and z_{max}, several comparisons were made with the steady state solu-

TABLE 2. Values of Q^* From the Finite Element Program, Compared to Results From *Wooding* [1968] and *Weir* [1987] Using α From the *Gardner* [1958] Conductivity Function

	r_{max}/r_0						
αr_0	5	10	15	25	50	Wooding	Weir
0.01	...	...	...	0.0354	0.0417	0.0403	0.0408
				(0.0417)	(0.0423)		
0.1	...	0.425	0.444	0.454	...	0.431	0.451
	...	(0.426)	(0.454)	(0.456)	...		
1.0	7.41	7.46	...	...	...	7.14	7.86*
	(7.44)	(7.46)	...	...	...		

Results in parentheses are from a type 1 boundary condition at $r = r_{max}$; others are for a "no-flow" boundary.
*Beyond recommended maximum of $\alpha r_0 = 0.8$.

tions of *Wooding* [1968] and *Weir* [1987] which assume (6). Calculations were performed with $\alpha r_0 = 0.01$, 0.1, and 1, chosen to correspond to α roughly in the range of 0.5–50 m^{-1} and r_0 from 0.05–0.1 m. The actual calculations using the program "Disc" were with $r_0 = 1$ and α chosen appropriately. Most of the tests were with a total of 2500 nodes (50 × 50). For the r direction, 15 nodes were equally spaced on the disc, with the remaining 35 nodes located by following an arithmetic progression to r_{max}. The smallest z interval was the same as the smallest r interval ($r_0/25$); the other 48 nodes were at intervals following an arithmetic progression to $z = z_{max} = r_{max}$. The resulting dimensionless $Q^* = \alpha^2 Q/K_s$ are given in Table 2 for several r_{max}/r_0 ratios. The Q^* increase as αr_0 gets larger. Also, as r_{max}/r_0 gets larger, the Q^* tends to increase toward a maximum for a given αr_0. Generally, as r_{max} increases, results for the no-flow condition at $r = r_{max}$ are equivalent to those for a head-specified boundary condition with h at $r = r_{max}$ chosen at -100 m.

Wooding [1968] and *Weir* [1987] used a dimensionless flux given by

$$F = \frac{Q}{\lambda_c K_{wet} r_0} \tag{24}$$

or

$$F = \frac{Q^*}{\alpha^2 \lambda_c r_0} \tag{25}$$

Wooding's relationship (5) is equivalent to $F = F_W$, with

$$F_W = 4 + 2\pi a \tag{26}$$

where

$$a = 0.5 r_0 \lambda_c^{-1} \tag{27}$$

For small a, values of F_W approach 4, which would be the value if capillarity totally dominated over gravity. For large a the result approaches that for one-dimensional flow.

On careful examination, *Weir* [1987] found F_W to be somewhat inaccurate for small a. He suggested an alternative approximation when $a < 0.4$ as

$$F_W^* = \frac{4\pi \sin^2 a}{a\pi \sin(a) \cos(a) + 2a \sin^2(a) \ln(a) - 1.073a^3} \tag{28}$$

Results based on the *Wooding* [1968] and *Weir* [1987] relationships are presented in Table 2. Generally, they agree with the finite element values. The largest value of $\alpha r_0 = 1$ corresponds to $a = 0.5$ which is just above Weir's cutoff of $a = 0.4$. Weir pointed out that F_W is too small for small αr_0 and that the finite element values are in line with his numerical results (especially his Table 1). On the basis of the results, values of r_{max}/r_0 for other calculations were chosen as 25–50 for smaller values of αr_0 (and $\lambda_c^{-1} r_0$) and 5–15 for the larger values.

The choice of $\lambda_c = \alpha^{-1}$, where α is from the *Gardner* [1958] conductivity function (6), has been put forth as a generalization, allowing (5) [*Wooding*, 1968] and related formulas to be applied for other conductivity functions. We now directly compare steady state inflow rates calculated using *van Genuchten*'s [1980] characteristic curve with those based on λ_c. Values of m for *van Genuchten* [1980] functions of 0.3, 0.5, 0.7, and 0.9 were chosen. These represent a wide range of conditions (m is always between 0 and 1). The values of the integral $\alpha\lambda_c$, where now α is from the van Genuchten relationships, are a function only of m, αh_{dry}, and αh_{wet}. The $\alpha\lambda_c$ values for $\alpha h_{dry} = -\infty$ and $\alpha h_{wet} = 0$ were calculated by direct integration. Additional values are given by *Warrick and Broadbridge* [1992]. The values increase from 0.188 at $m = 0.3$ to 0.875 at $m = 0.9$, indicating that the corresponding α for *Gardner*'s [1958] equation are larger than those from van Genuchten's relationship for the same λ_c (see Table 3).

Calculations for Q^* were performed with $\alpha r_0 = 0.01$, 0.1, and 1, where now α is from the *van Genuchten* [1980] relationship (Table 1). The Q^* behaves similarly to those for the *Gardner* [1958] model. Smaller αr_0 values correspond to smaller Q^* values. As m gets smaller, Q^* also tends to decrease. This is consistent with the notion that Q^* decreases as $\lambda_c^{-1} r_0$ gets smaller.

Values of F from (24) are presented in Table 3 to facilitate ease of comparisons with F_w (Wooding) and F_w^* (Weir). Also, ratios of F/F_w for $a > 0.4$ and F/F_w^* for $a < 0.4$ are given. If the integral relationship (3) totally captured the steady infiltration for the van Genuchten conductivity function, the F/F_w or F/F_w^* would be 1. As it is, the values tend to be between 0.80 and 1.4, with the smallest ratios corresponding to the smaller m. For the moderate values of $m = 0.5$ and 0.7, F/F_W or F/F_w^* is between 0.98 and 1.2.

The steady state flow rate due to capillary forces, derived from (5), is

TABLE 3. Dimensionless Disc Infiltration Rates for Steady State Conditions Using van Genuchten Functions

m	$\alpha\lambda_c$	αr_0	Q^*	$\alpha^2\lambda_c r_0$	F	a	F_W	F_W^*	F/F_W or F/F_W^*	Q_D/Q_{ss}
0.3	0.188	0.01	0.00751	0.00188	3.84	0.0266	4.17	4.30	0.928	1.00
		0.1	0.0936	0.0188	5.28	0.266	5.67	6.07	0.820	1.24
		1.0	3.17	0.188	17.7	2.66	20.7	n.a.	0.814	4.22
0.5	0.405	0.01	0.0166	0.00405	4.25	0.0124	4.08	4.16	0.987	1.02
		0.1	0.203	0.0405	5.14	0.123	4.78	5.08	1.02	1.25
		1.0	4.86	0.405	11.9	1.23	11.8	n.a.	1.02	3.00
0.7	0.637	0.01	0.0267	0.00637	4.33	0.00785	4.05	4.11	1.02	1.05
		0.1	0.328	0.0637	5.27	0.0785	4.49	4.74	1.08	1.29
		1.0	7.04	0.637	10.8	0.785	8.93	n.a.	1.23	2.76
0.9	0.875	0.01	0.0372	0.00875	4.47	0.00571	4.04	4.09	1.04	1.06
		0.1	0.466	0.0876	5.44	0.0571	4.36	4.57	1.17	1.33
		1.0	9.33	0.875	8.57	0.571	7.59	n.a.	1.40	2.67

Here n.a. means nonapplicable.

$$Q_D = 4K_{\text{wet}}\lambda_c r_0 \tag{29}$$

which is valid for all conductivity functions for $K_{\text{wet}} \gg K_{\text{dry}}$. A comparison of (24) with (29) reveals that $F \geq 4$ for all hydraulic conductivity functions and will be equal to 4 if gravity is neglected. Values of Q/Q_D are included as the final column of Table 3. For small αr_0 values, Q approaches Q_D, and the ratio is close to one.

Example 2: Infiltration at Small Times

Simulations for small-time infiltration were performed using the simple one-dimensional expression (2). Also, simulations of the constant D were performed using approximation (16) as well as the nonlinear numerical model. Soil parameters are given in Table 4. For the first comparison consider the hypothetical loam of *Warrick et al.* [1985] with $r_0 = 0.1$ m. The sorptivity value was determined directly to be $S = 1.21 \times 10^{-3}$ m s$^{-0.5}$. An average value of D is defined as

$$D_{\text{avg}} = (\theta_{\text{wet}} - \theta_{\text{dry}})^{-1} \int_{\text{dry}}^{\text{wet}} D(\theta)\, d\theta \tag{30}$$

(Other average values of D could also be used, for example, the weighted mean of *Philip* [1969].) From (3) and the basic definition $D = K(dh/d\theta)$, D_{avg} is

$$D_{\text{avg}} = (\theta_{\text{wet}} - \theta_{\text{dry}})^{-1}(K_{\text{wet}} - K_{\text{dry}})\lambda_c \tag{31}$$

From Table 3 we have $\lambda_c\alpha = 0.405$. Thus if $K_{\text{wet}} = K_{\text{sat}}$ and $K_{\text{dry}} = 0$, then D_{avg} is 6.94×10^{-6} m^2 s^{-1}.

In Figure 2*a* the flow value per unit area (I) is plotted as a function of $t^{1/2}$ using the one-dimensional relationship (1), the *Chu et al.* [1975] correction from (19), and "linear diffusion" from (17). Also, the finite element result is plotted on Figure 2*a*. The finite element mesh was defined by 2500 nodes, 50 in the r and 50 in the z direction. In the r direction, 15 were equally spaced on the 0.1-m disc and the other 35 defined by arithmetic progression to $r_{\max} = 10r_0 = 1$. The minimum z spacing was the same as for the disc node spacing (0.1/15 m). and the other 48 depths increased as an arithmetic progression to $z_{\max} = 10r_0 = 1$ m. The values of Δt began at 0.25 s and were increased geometrically to 300 s. Results for the numerical solution start out very close to the three analytical curves. The values quickly exceed the one-dimensional solution but follow the two-term relationship (19) a bit longer. For larger values the results exceed the three-term linear diffusion results (17) but generally follow much closer.

As the second comparison for small time we choose the silt loam of *Parker et al.* [1985] for which the "Gardner-Russo" parameters from *Russo* [1988] are listed in Table 4. Calculations are for two h_{wet} values; 0 and -0.15 m. Corresponding S values are 1.71×10^{-3} and 1.42×10^{-3} m s$^{-0.5}$, respectively. As $\lambda_c = \alpha^{-1}$, D is easily found as 2.69×10^{-5} and 1.89×10^{-5} m^2 s^{-1} from (31). Results are plotted as before, for both $h_{\text{wet}} = 0$ and $h_{\text{wet}} = -0.15$, in Figure 3*a*. The early time values are very close to all of the theoretical curves. The finite element quickly rises above the one-dimensional results. follows along the *Chu et al.* [1975] correction until about $t^{0.5} = 10$, and then remains reasonably close to the three-term "diffusion" results for the duration of the plot. The plot also demonstrates that a smaller infiltration rate occurs for the tension ($h_{\text{wet}} = -0.15$ m) conditions.

The final short-term plot (Figure 4) shows results for the

TABLE 4. Soil Parameters Used

	Reference	Functions	α, m^{-1}	m	θ_s	θ_r	K_s, m s^{-1}
Loam	*Warrick et al.* [1985]	VG	1.0	0.5	0.45	0.1	6×10^{-6}
Silt loam	*Parker et al.* [1988] and *Russo* [1988]	GR	2.38	5.14	0.388	0.154	1.5×10^{-5}
Yolo light clay	*Moore* [1939] and *Warrick et al.* [1985]	VG	1.5	0.5	0.495	0.124	1.2×10^{-7}

VG, *van Genuchten* [1980]; GR, *Gordner* [1958] and *Russo* [1988].

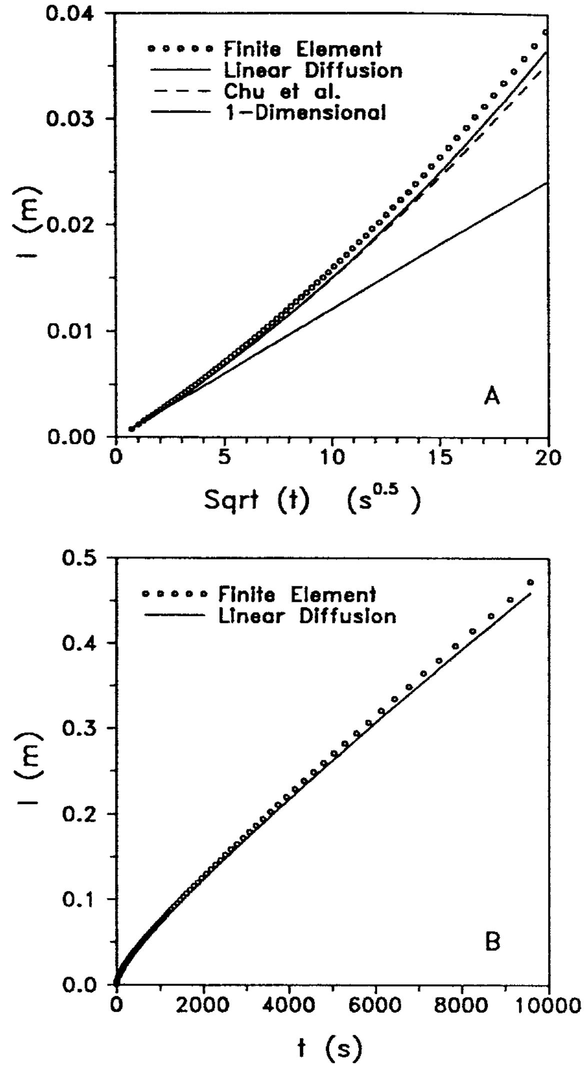

Fig. 2. (*a*) Small-time infiltration plot. (*b*) Intermediate-time infiltration plot for the loam of Table 4 (r_0 = 0.1 m).

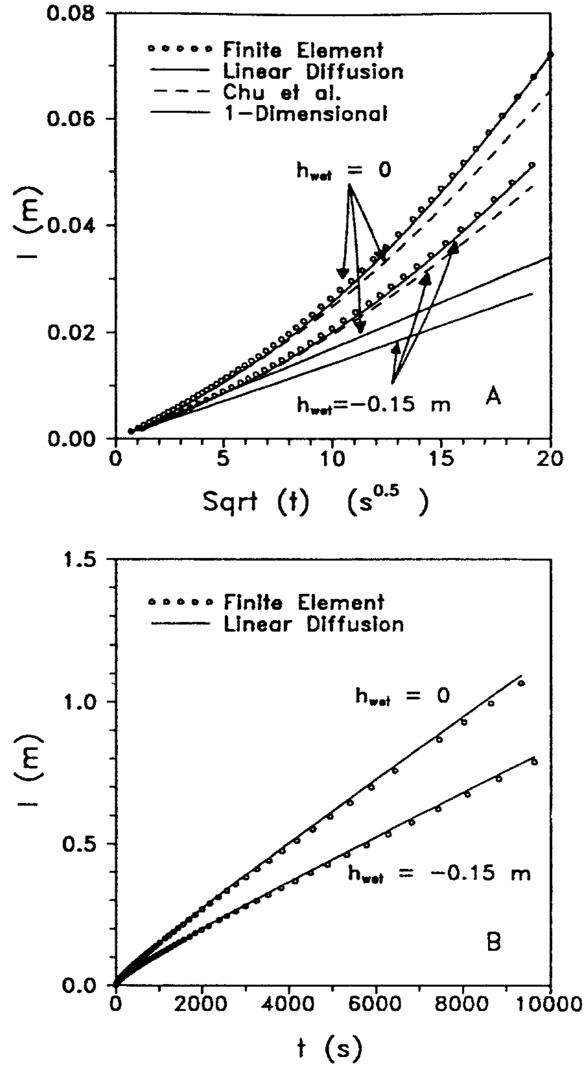

Fig. 3. (*a*) Small-time infiltration. (*b*) Intermediate-time plots for silt loam of *Parker et al.* [1985] and *Russo* [1988].

Yolo light clay. The S and D were found as 1.54×10^{-4} m $s^{-0.5}$ and 8.95×10^{-8} m^2 s^{-1}. For this case, infiltration is slower, and all of the curves are reasonably close together. However, for larger times the results from the finite element solution are considerably closer to the linear diffusion results. For all intents and purposes the three-term and two-term "*Chu et al.*" [1975] solution are the same at this scale and are plotted as one curve.

Characteristic times t_{geom} and t_{grav} were proposed by *Philip* [1969] to estimate when the three-dimensional geometry or gravity would dominate the flow process. These may be written as

$$t_{\text{geom}} = r_0^2/D \tag{32}$$

$$t_{\text{grav}} = \left(\frac{S}{K_{\text{wet}}}\right)^2 \tag{33}$$

The extended period over which the one-dimensional approximation for the Yolo holds is consistent with the large values of t_{geom} and t_{grav}. For t_{geom} the value is about 30 hours compared to less than 1 hour for the other soils.

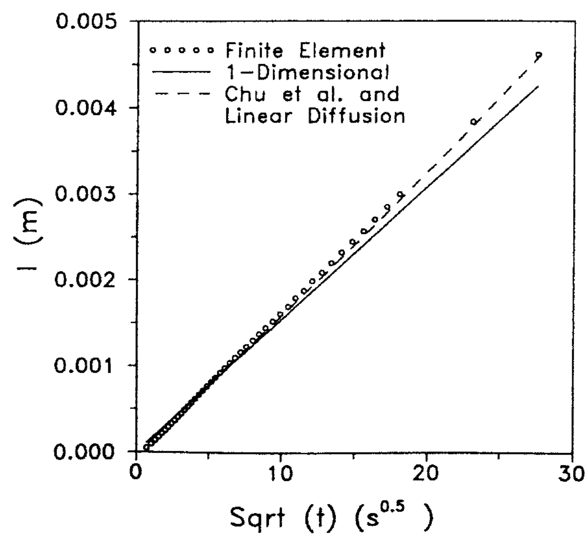

Fig. 4. Small-time infiltration plot for the Yolo light clay.

Example 3: Medium- and Large-Time Results

The results using the linear diffusion expression (16) and the finite element were compared in Figure 2*b* for an "intermediate" time period taken arbitrarily as 0–10,000 s. The comparison shows reasonable agreement even though gravity is not included in the linear diffusion term. For a longer time period (0.01 to 10 hours) the ratio of the flow rate Q and the steady flow rate Q_{ss} is given in Figure 5. The decay is quite slow but eventually Q reaches 1.2 and 1.05 of the final Q_{ss} after 0.5 and 13.5 hours, respectively.

For this example, t_{geom} is 0.4 hours and t_{grav} is 11.3 hours. Also, we can compare these results to Figure 4 of *Pullan* [1988]. For our case the time to reach 1.05 of the final flux is approximately

$$t = 0.25\ t^*_{\text{grav}} \quad (34)$$

$$t^*_{\text{grav}} = 4\lambda_c^2/(\pi D) \quad (35)$$

Using $D = 6.94 \times 10^{-6}$ m^2 s^{-1} and $\lambda_c = 0.405$ gives $t^*_{\text{grav}} =$ 8.4 hours or t above is 2.1 hours. This value was checked also by the finite element program, resulting in a value of 2 hours for Q to reach 1.05 Q_{ss} for the linearized case. For 1.2 Q_{ss} the time was approximately 0.3 hours. These values are repeated in Table 5.

Similar nonlinear finite element results are found for the silt loam of *Parker et al.* [1985] (Table 4). For the source potential at both 0 and −0.15 m the nonlinear solution was tracked by the linear diffusion model, which does not include gravity from 0–10,000 s (see Figure 3*b*). Results for longer times are given as Figure 5 and in Table 5. For the nonlinear case, times of 4.4 and 0.4 hours were required to reach 1.05 and 1.2 times the final intake rate. The limiting intake rate 6.61 compares to values, based on (25) and (27), of 6.35 and 6.74, respectively. For the linearized case the times were approximately 0.6 and 0.1 hours. *Pullan*'s [1988] results for 1.05 from the final rate (his Figure 2) is approximately 0.25 t^*_{grav} again, where now t^*_{grav} = 2.3 hours, which is in agreement with the linear, finite element results. For the linearized case the times were approximately 0.6 and 0.1 hours.

Comparisons can also be made for the time t_Q to reach 1.05 Q_{ss} on the basis of a surface, hemispherical source [after *Philip*, 1986, Figure 4], also for linearized conditions. For both of our last two examples, $a = 0.5 r_0 \lambda_c^{-1} \approx 0.12$. From Figure 4 of Philip we read $t_Q \approx 0.4\ t_{\text{grav}}$. Thus t_Q is (0.4)(11.2) ≈ 5 hours and (0.4)(3.6) ≈ 1.5 hours. These fall between the linear and nonlinear simulations both for the loam and silt loam. (If "a" is chosen so as to give the same surface area for the hemisphere as for the disc, then t_Q will be reduced to about 0.3 t_{grav}).

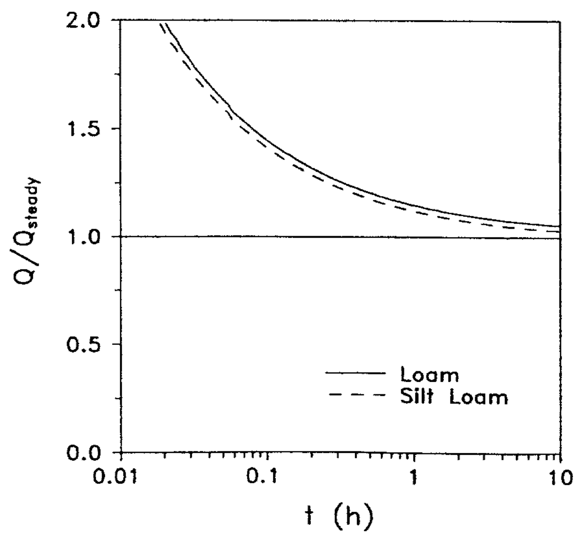

Fig. 5. Decay of flow rate for the loam and silt loam of Table 4.

TABLE 5. Times to Reach Within 1.05 and 1.2 Times the Steady Infiltration Rate

	$t_{1.05}$, hours	$t_{1.2}$, hours
Loam ($Q_{ss}/[\pi r_0^2 K_{wet}]$ = 6.45, t_{geom} = 0.4 hours, t_{grav} = 11.2 hours)*		
Nonlinear	13.5	0.5
Linear (λ_c = 0.405 m, $D = 6.94 \times 10^{-6}$ m^2 s^{-1})		
Finite element	2.0	0.3
Pullan [1988]	2.1	···
Silt Loam† ($Q_{ss}/[\pi r_0^2 K_{wet}]$ = 6.61, t_{geom} = 0.1 hours, t_{grav} = 3.6 hours)		
Nonlinear	4.4	0.4
Linear ($D = 2.69 \times 10^{-5}$ m^2 s^{-1})		
Finite element	0.6	0.1
Pullan [1988]	0.6	···

*[*Warrick et al.*, 1985].
†[*Parker et al.*, 1985; *Russo*, 1988].

DISCUSSION

Some generalized solutions for disc infiltrometers have been examined. In particular, we have assessed the applicability of the *Wooding* [1968] analysis for steady state rates for an alternative hydraulic function as well as the application of the small-time one-dimensional diffusion relationships and predictions of times necessary to approach steady state.

The steady state flux rate calculations for the *van Genuchten* [1980] conductivity function were from 80–140% of those calculated using the same λ_c in the *Wooding* [1968] and *Weir* [1987] relationships. Matching λ_c is nearly equivalent to matching S, as done by *Russo et al.* [1991]. Whether this is sufficiently accurate depends on the application, but certainly it would be for many purposes. For smaller discs the values of steady flow rate Q for the van Genuchten function tends to be smaller, and conversely, for larger αr_0 the values are as large or larger. As r_0 becomes small, gravity becomes less of a factor and Wooding's relationship and all conductivity relationships approach the flow rate $Q_D = 4K_{\text{wet}}\lambda_c r_0$.

The short-term analytical refinements to the one-dimensional solution are from a recent approximation for diffusion from a disc by *Warrick et al.* [1992]. This is based on the solution by *Chu et al.* [1975] for a two-dimensional edge effect. Of the three cases considered, the refinement brought the analytical results much closer to the simulated disc results than the one-dimensional solution alone. This would allow a longer range of applicability for small-time results, in fact, the linear approximation tended to be reasonable for time as large as 3 hours in the two cases considered. The correction terms are easily applied and can be expressed in terms of only the sorptivity and initial water supply contents by (4), (19), and (31):

$$\frac{Q}{\pi r_0^2} \approx 0.5St^{-0.5} + \frac{0.885b^{0.5}S^2}{r_0(\theta_{\text{wet}} - \theta_{\text{dry}})^2} \quad (36)$$

Of course, the soil, boundary conditions, and initial conditions control the time to approach steady state flow. For the two examples considered, 4 and 13 hours were required to approach 1.05 of the final infiltration value, which is the same order of magnitude as the t_{grav} used by *Philip* [1969] of 3.6 and 11.2 hours. These values are considerably longer than the 0.6 and 2 hours that were the approximate time for the linearized solution with gravity. For the numerical examples here it would be extremely difficult to ascertain when true steady state was reached without knowing the limiting values (see Figures 2*b* and 3*b*). This suggests that the theoretical time for approaching steady state may be longer than typical experimental times reported of approximately 0.2–2 hours [*White and Sully*, 1988; *Smettem and Clothier*, 1989; *Ankeny et al.*, 1990; *Thony et al.*, 1991]. However, the reported experimental times are consistent with the observation that S tends to be smaller in the field than for repacked laboratory columns; hence a smaller t_{grav} for the field would follow. *Sully and White* [1987] calculated the t_Q of *Philip* [1986], the time to approach 1.05 of the steady state value, for 17 scattered field sites. The geometric mean value was less than two hours, but the largest was 31 hours.

The emphasis has been to compare simplified versus comprehensive calculations. The next apparent step is to use this information to design more complete schemes for parameter identification.

APPENDIX: FLOW RATE FOR LINEAR DIFFUSION FROM A DISC SOURCE

Warrick et al. [1992] present an approximation of the time-dependent flow rate for linear diffusion from a disc source. The approximation is presented graphically in Figure 6 but is difficult to present algebraically. Consequently, a second function of the form

$$Q = A + BT^{-0.5} + C\exp(-ET) \quad (37)$$

was used. Values of A, B, and C are chosen to give the correct large-time solution and to satisfy (18). The large-time solution may be given from (28) or equivalently by

$$Q_D = 2\pi^{0.5}SD^{0.5}r_0 \quad (38)$$

Thus only E was best fit, giving $E = 4.01$ in (37). The two relationships are in excellent agreement, as shown in Figure 6, with a coefficient of determination above 0.999 for the points shown. The slight "dip" in the old approximation as $T = 1$ is close to where two branches were matched by Warrick et al. The Q values at the matching points were continuous but the slopes were not.

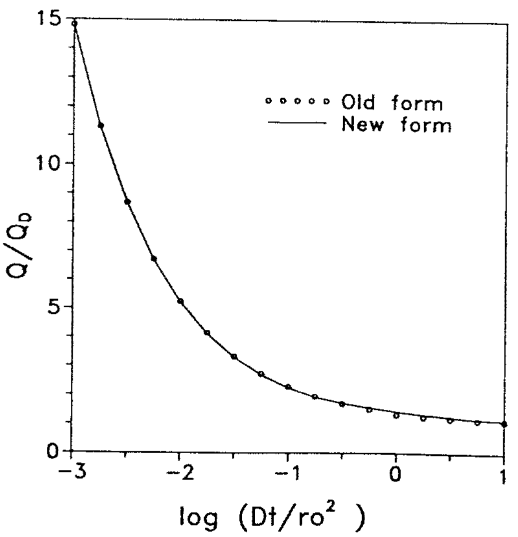

Fig. 6. A comparison of the old form of Q/Q_D [*Warrick et al.*, 1992] to the new approximation (16).

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